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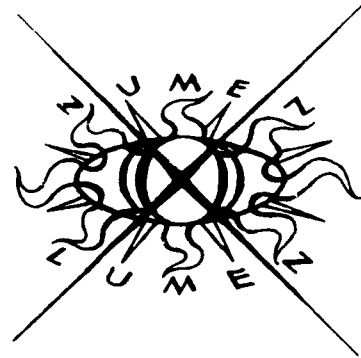
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ON POLYNOMIAL SPLINE FUNCTIONS ON THE
CIRCLE II. MONOSPLINES AND QUADRATURE
FORMULAE

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13. ABSTRACT

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ON POLYNOMIAL SPLINE FUNCTIONS ON THE CIRCLE II.

MONOSPLINES AND QUADRATURE FORMULAE

I. J. Schoenberg

Introduction

In [4] we have studied the interpolatory properties of the class $\mathcal{S}_{m,k}$ of polynomial spline functions on the circle that were first discussed by Ahlberg, Nilson and Walsh (see [1], also for further references). In the present second paper we discuss quadrature formulae of the form

$$(1) \qquad \int_U f(z)dz = \sum_0^{k-1} C_j f(\omega^j) + Rf \quad ,$$

where U is the unit circle $|z| = 1$ described counter-clockwise and $\omega = \exp(2\pi i/k)$. This problem requires a discussion of monosplines and will suggest several classes of spline functions and monosplines having noteworthy properties. Free use is made of the results of [4], in particular of B-splines and their Fourier series expansions.

1. **Monosplines and quadrature formulae.** Let k and m be positive integers. Our objective is to derive quadrature formulae of the form (1) which are to be exact for polynomials of degree not exceeding $m-1$, i.e.

$$(1.1) \qquad Rf = 0 \quad \text{if} \quad f \in \pi_{m-1} \quad .$$

This property is equivalent to the relations

$$(1.2) \qquad \int_U z^\nu dz = 0 = \sum_{j=0}^{k-1} C_j \omega^{j\nu} , \qquad (\nu = 0,1,\ldots,m-1) \quad ,$$

and these show that the quadrature formula (1) has the property (1.1) if and only if

(1.3)   the polynomial $\sum\limits_{0}^{k-1} C_j x^j$ vanishes if $x = 1, \omega, \ldots, \omega^{m-1}$.

We conclude the following: 1. If $m \geq k$ then the polynomial (1.3) vanishes identically, hence $C_j = 0$ for all j . 2. If $m < k$ then the polynomial (1.3) still depends on $k-m$ arbitrary linear parameters. Avoiding trivial cases we shall assume that

(1.4) $1 \leqq m < k$.

Expanding the binomials appearing in (1.5) we see that the condition (1.3) is equivalent to the validity of the identity in z

(1.5) $\sum\limits_{0}^{k-1} C_j (z-\omega^j)^{m-1} = 0$.

At this point we reintroduce the class $\mathbf{S}_{m,k}$ of polynomial spline functions on U of degree m-1 having the ω^j as knots, and for a good reason. In fact the identity (1.5) is the precise condition for the existence of a function $S(z) \in \mathbf{S}_{m,k}$ such that

(1.6) C_j = the jump of $S^{(m-1)}(z)$ at $z = \omega^j$ $(j = 0, \ldots, k-1)$.

Indeed, if $S(z) = Q(z) \in \pi_{m-1}$ on the arc $(\omega^{k-1}, 1)$ then the function on U defined on the arcs $(\omega^{k-1}, 1), (1, \omega), \ldots, (\omega^{k-1}, 1)$ by the successive partial sums of the expression

$$Q(z) + \sum\limits_{0}^{k-1} C_j \frac{(z-\omega^j)^{m-1}}{(m-1)!}$$

is, in view of (1.5), a spline function S(z) with the property (1.6).

The connection of the spline function S(z) with the quadrature formula (1) is close as shown by

Lemma 1. **If we write**

$$(1.7) \qquad K(z) = \frac{1}{m!} z^m - S(z)$$

<u>and if</u>

$$(1.8) \qquad f(z) \in C^m(U)$$

<u>then we have the relation</u>

$$(1.9) \qquad \int_U f(z)\, dz = \sum_0^{k-1} C_j f(\omega^j) + (-1)^m \int_U K(z) f^{(m)}(z)\, dz \quad .$$

<u>Proof.</u> By (1.6) and (1.7) we find that

$$(1.10) \qquad - C_j = \text{the jump of } K^{(m-1)}(z) \text{ at } z = \omega^j \quad .$$

Integrations by parts show that

$$\int_U K(z) f^{(m)}(z)\, dz = - \int_U K'(z) f^{(m-1)}(z)\, dz = \ldots = (-1)^{m-1} \int_U K^{(m-1)}(z) f'(z)\, dz$$

a point beyond which we can not continue the process because of the discontinuities of $K^{(m-1)}(z)$. However, we may proceed as follows

$$\int_U K(z) f^{(m)}(z)\, dz = (-1)^{m-1} \sum_j \int_{\omega^j}^{\omega^{j+1}} K^{(m-1)}(z) f'(z)\, dz$$

$$= (-1)^{m-1} \sum_j \left\{ K^{(m-1)}(\omega_-^{j+1}) - K^{(m-1)}(\omega_+^j) f(\omega^j) - \int_{\omega^j}^{\omega^{j+1}} f(z)\, dK^{(m-1)}(z) \right\}$$

and using (1.10) we obtain

$$\int_U K(z) f^{(m)}(t)\, dz = (-1)^{m-1} \sum C_j f(\omega^j) + (-1)^m \int_U f(z)\, dz$$

because $d K^{(m-1)}(z) = dz$ within each of the arcs.

Using a term familiar from the real axis case (see e.g. [3]) we call $K(z)$ a <u>monospline</u> of degree m and denote their class by the symbol $\mathfrak{m}_{m,k}$. We may then summarize our results as follows: To every quadrature formula (1) with the property (1.1) corresponds a monospline $K(z)$ producing the identity (1.9). Conversely, to every monospline $K(z)$ corresponds a quadrature

formula (1), the C_j being described by (1.6). This correspondence is one-to-one up to an element of π_{m-1} that we may add to $K(z)$ without changing our results. Our discussion of the quadrature formula (1) will single out certain spline functions and monosplines for special study.

2. <u>The flower-shaped spline functions</u>. Let $S(z) \epsilon S_{m,k}$ and let us assume that it satisfies the functional equation

(2.1) $$S(z\omega) = \omega^r S(z) , \qquad (|z| = 1) ,$$

where r is an integer, $0 \leqq r \leqq k-1$. We could say that $S(z)$ is quasiperiodic, or r-quasiperiodic. However, in view of the rotational symmetry of the image of U by $w = S(z)$ we prefer to say that $S(z)$ is <u>flower-shaped</u> and also that $S(z)$ is an r-<u>flower</u> (see Fig. 1 in [4] showing the image of U by a 3-flower for $k = 5$, $m = 2$).

In terms of the B-spline $M_m(z)$ of [4] we have

<u>Lemma 2</u>. <u>For each</u> r, $0 \leqq r \leqq k-1$, <u>there is up to a constant factor a unique</u> r-<u>flower given by the formula</u>

(2.2) $$S(z) = \sum_{j=0}^{k-1} \omega^{jr} M_m(z\omega^{-j}) .$$

<u>Proof</u>: By [4, Lemma 1] we may write

$$S(z) = \sum c_j M_m(z\omega^{-j}) ,$$

and (2.1) gives $\sum c_j M_m(z\omega^{1-j}) = \sum c_j \omega^r M_m(z\omega^{-j})$ or $\sum c_{j+1} M_m(z\omega^{-j}) = \sum c_j \omega^r M_m(z\omega^{-j})$, whence $c_{j+1} = c_j \omega^r$ and finally $c_j = c_0 \omega^{jr}$. Setting $c_0 = 1$ we obtain (2.2). Conversely, it is easily established that (2.2) satisfies (2.1) and Lemma 2 is established.

The Fourier expansions of r-flowers are described by

<u>Lemma 3</u>. <u>If</u> $S(z)$ <u>is an</u> r-<u>flower then, up to a constant factor</u>,

(2.3) $$S(z) = z^r \quad \text{if} \quad r = 0, 1, \ldots, m-1 \quad ,$$

<u>and</u>

(2.4) $$S(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+r)(ks+r-1)\ldots(ks+r-m+1)} z^{ks+r}$$

$$\underline{\text{if}} \quad r = m, m+1, \ldots, k-1 \quad .$$

<u>Proof</u>: We recall [4, formula (2.16)] that

(2.5) $$M_m(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\nu b_{\nu-1} \ldots b_{\nu-m+1} z^\nu$$

where

(2.6) $$b_\nu = \frac{1-\omega^{-\nu}}{\nu} \quad \text{if} \quad \nu \neq 0, \quad b_0 = \frac{2\pi i}{k} \quad .$$

From (2.2) and (2.5)

$$S(z) = \frac{1}{2\pi i} \sum_\nu b_\nu \ldots b_{\nu-m+1} \left(\sum_{j=0}^{k-1} \omega^{jr} \omega^{-j\nu} \right) z^\nu$$

or

(2.7) $$S(z) = \frac{k}{2\pi i} \sum_\nu b_\nu b_{\nu-1} \ldots b_{\nu-m+1} z^\nu \quad ,$$

where the summation is over all $\nu \equiv r \pmod{k}$, hence $\nu = ks+r$. By (2.6)

we see that $b_\nu = 0$ if ν is a nonvanishing multiple of k. If $0 \leq r \leq m-1$

then (2.7) is seen to reduce to the single term corresponding to $s = 0$ or

$\nu = r$ and this is indeed proportional to (2.3). Also directly it is clear that

$z^r \in \mathcal{S}_{m,k}$ and that z^r is an r-flower.

If $m \leq r \leq k-1$, we may discard on the right side of (2.7) the non-

vanishing factor

$$\frac{k}{2\pi i} (1-\omega^{-r})(1-\omega^{-r+1}) \ldots (1-\omega^{-r+m-1})$$

and we obtain the expansion (2.4).

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3. __The monospline of least L_2-norm.__   We have already used the fact

that

(3.1)
$$S(z) = \sum_{0}^{k-1} c_j M_m(z\,\omega^{-j})$$

represents the most general element of $\mathbf{S}_{m,k}$.   Using (2.5) we find its

Fourier expansion to be

$$S(z) = \frac{1}{2\pi i} \sum_{\nu} b_\nu b_{\nu-1} \cdots b_{\nu-m+1} z^\nu \sum_{j=0}^{k-1} c_j \omega^{-\nu j} \quad .$$

However

(3.2)
$$\eta_\nu = \sum_{j=0}^{k-1} c_j \omega^{-\nu j}$$

in the finite Fourier series representation of an arbitrary periodic sequence

(η_ν) of period k .   This establishes

> __Lemma 4.__   __The most general element of__ $\mathbf{S}_{m,k}$ __is given by__

(3.3)
$$S(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\nu b_{\nu-1} \cdots b_{\nu-m+1} \eta_\nu z^\nu$$

__where__ (η_ν) __is an arbitrary periodic sequence of numbers of period__ k .

It is now easy to determine the monospline

(3.4)
$$K(z) = \frac{1}{m!} z^m - S(z) , \qquad S(z) \in \mathbf{S}_{m,k}$$

having least L_2-norm, or

(3.5)
$$\|K\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |K(z)|^2 \, d\theta \right)^{\frac{1}{2}} = \text{minimum} , \qquad (z = e^{i\theta}) \quad .$$

Indeed, by Lemma 4 we may write

$$K(z) = \frac{1}{m!} z^m - \frac{1}{2\pi i} \sum_{\nu} b_\nu \cdots b_{\nu-m+1} z^\nu \eta_\nu$$

$$= \left(\frac{1}{m!} - \frac{1}{2\pi i} b_m \cdots b_1 \eta_m \right) z^m - \frac{1}{2\pi i} \sum_{\nu \neq m} b_\nu \cdots b_{\nu-m+1} \eta_\nu z^\nu$$

and using Parseval's relation we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \left| K(z) \right|^2 d\theta = \left| \frac{1}{m!} - \frac{1}{2\pi i} b_m \ldots b_1 \eta_m \right|^2 + \frac{\left| \eta_m \right|^2}{4\pi^2} \sum_{\substack{\nu \equiv m(k) \\ \nu \neq m}} \left| b_\nu \ldots b_{\nu-m+1} \right|^2$$

(3.6)
$$+ \frac{1}{4\pi^2} \sum_{\substack{\ell=0 \\ \ell \neq m}}^{k-1} \left| \eta_\ell \right|^2 \sum_{\nu \equiv \ell(k)} \left| b_\nu \ldots b_{\nu-m+1} \right|^2 \quad .$$

Observe that by (2.6)

$$\sum_{\nu \equiv \ell(k)} \left| b_\nu \ldots b_{\nu-m+1} \right|^2 > 0 \qquad \text{for all integer } \ell \quad .$$

It follows that in minimizing the right side of (3.6) we must have

(3.7) $\qquad \eta_\ell = 0 \quad$ if $\quad \ell = 0, 1, \ldots, m-1, m+1, \ldots, k-1$

Therefore the search reduces to the much narrower class of monosplines of the form

(3.8) $\qquad K(z) = \frac{1}{m!} z^m - \frac{1}{2\pi i} \eta_m \sum_{\nu \equiv m(k)} b_\nu \ldots b_{\nu-m+1} z^\nu \quad .$

In fact, by Lemma 3, we see that (3.8) describes the most general mono-spline $K(z)$ satisfying the relation

(3.9) $\qquad\qquad K(z\omega) = \omega^m K(z) \quad .$

Setting

$$\lambda = \frac{1}{2\pi i} (1-\omega^{-m})(1-\omega^{-m+1}) \ldots (1-\omega^{-1}) \eta_m$$

we may write (3.8) as

(3.10) $\qquad K(z) = \frac{1}{m!} z^m - \lambda \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+z)\ldots(ks+m)} z^{ks+m} \quad ,$

and (3.6) becomes

(3.11) $\quad (m!)^2 (\| K \|_2)^2 = \left| 1-\lambda \right|^2 + \left| \lambda \right|^2 \sum_{s \neq 0} \frac{(m!)^2}{(ks+1)^2(ks+2)^2 \ldots (ks+m)^2} \quad ,$

and we are to determine the value of λ that minimizes the right side. The

inequalities $|1-\lambda| \geqq |1-\operatorname{Re}\lambda|$, $|\lambda| \geqq |\operatorname{Re}\lambda|$ show that this value of λ

must be real. Writing

$$(3.12) \qquad c = \sum_{s \neq 0} \frac{(m!)^2}{(ks+1)^2 \dots (ks+m)^2}$$

we see from the identity

$$(1-\lambda)^2 + c\lambda^2 = (1+c)(\lambda - \frac{1}{1+c})^2 + \frac{c}{1+c} \quad , \qquad (c > 0) \quad ,$$

that the least value of the left-hand side is $= c/(1+c)$ and that it is assumed

if $\lambda = \lambda_* = 1/(1+c)$. By (3.12) we see that the right side of (3.11) is least

if λ equals

$$(3.13) \qquad \lambda_* = \left(\sum_{s=-\infty}^{\infty} \frac{(m!)^2}{(ks+1)^2 (ks+2)^2 \dots (ks+m)^2} \right)^{-1} \quad .$$

The least value of the right side of (3.11) we now find to be equal $c/(1+c) =$

$(\lambda_*^{-1} - 1)\lambda_* = 1 - \lambda_*$.

We have therefore established

Theorem 1. The unique monospline (3.4) of least L_2-norm is

$$(3.14) \qquad K_*(z) = \frac{1}{m!} z^m - \lambda_* S_m(z)$$

where

$$(3.15) \qquad S_m(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2) \dots (ks+m)} z^{ks+m} \quad ,$$

while λ_* is given by (3.13). The value of the least L_2-norm is

$$(3.16) \qquad \|K_*\|_2 = \frac{1}{m!} (1 - \lambda_*)^{\frac{1}{2}} \quad .$$

4. The best quadrature formula (1.9). We obtain the best quadrature

formula (1.9), by the definition of the term "best" introduced by A. Sard [2],

if we choose in (1.9) $K(z) = K_*(z)$, and there remains to determine the

coefficients C_j according to (1.10). However, we have already noticed that $K_*(z)$ satisfies the relation (3.9) which by differentiations yields $K_*^{(\nu)}(z\omega) = \omega^{m-\nu} K_*(z)$ and in particular

$$K_*^{(m-1)}(z\omega) = \omega K_*^{(m-1)}(z) \quad .$$

By (1.10) we see that

$$(4.1) \qquad\qquad C_j = \omega^j C_0$$

and it therefore suffices to determine the jump $\Delta_0 = -C_0$ of $K^{(m-1)}(z)$ at $z = 1$. It is easily obtained as follows.

Differentiating the Fourier series (3.14) $m-1$ times we get

$$(4.2) \qquad\qquad K_*^{(m-1)}(z) = z - \lambda_* \sum_s \frac{1}{ks+1} z^{ks+1} \quad .$$

Its sum is, as we know, a step function with discontinuities at the ω^j. To determine its jump Δ_0 at $z = 1$, we consider the function $\varphi(\theta)$ of period 2π such that

$$(4.3) \qquad\qquad \varphi(\theta) = \tfrac{1}{2}(\pi - \theta) \quad \text{if} \quad 0 < \theta < 2\pi \quad .$$

Defining also $\varphi(0) = 0$, we have the familiar expansion

$$(4.4) \qquad \varphi(\theta) = \sum_1^\infty \frac{\sin \nu\theta}{\nu} = \frac{1}{2i} \sum_{s \neq 0} \frac{1}{s} e^{is\theta} \quad \text{for all real } \theta \ ,$$

from which we derive that

$$\frac{1}{k} \varphi(k\theta) e^{i\theta} = \frac{1}{2i} \sum_{s \neq 0} \frac{1}{ks} z^{ks+1} \ , \qquad (z = e^{i\theta}) \quad .$$

Using (4.2) we find the expansion

$$(4.5) \quad K_*^{(m-1)}(e^{i\theta}) + 2i\lambda_* \frac{1}{k} \varphi(k\theta) e^{i\theta} = (1-\lambda_*) z + \lambda_* \sum_{s \neq 0} \frac{1}{ks(ks+1)} z^{ks+1}$$

which is seen to converge absolutely and uniformly on U. We conclude that

the left side belongs to $C(U)$. The jump of the function $\varphi(\theta)$ being $= \pi$, at $\theta = 0$, by (4.3), we conclude that the jump $2i\lambda_* k^{-1}\pi$ of the second term of the left side of (4.5) cancels the jump Δ_0 of the first term. Therefore

$$(4.6) \qquad \Delta_0 = -\frac{2\pi i}{k}\lambda_* \quad .$$

Using the relations (4.1) we obtain

Theorem 2. Let $1 \leqq m < k$. Among all quadrature formulae of the form (1) and having the property (1.1), the best is the quadrature formula

$$(4.7) \qquad \int\limits_z f(z)dz = \lambda_* 2\pi i k^{-1} \sum_{j=0}^{k-1} \omega^j f(\omega^j) + (-1)^m \int\limits_U K_*(z)f^{(m)}(z)dz$$

where $K_*(z)$ is the monospline of Theorem 1, while the positive fraction λ_* is defined by (3.13) .

Theorem 2 does not state or imply that the familiar average (or Riemann sum)

$$(4.8) \qquad A = 2\pi i k^{-1} \sum_0^{k-1} \omega^j f(\omega^j)$$

is not as good an approximation of the integral as the modified approximation $\lambda_* A$ appearing in (4.7) . However, Theorem 2 does state that $\lambda_* A$ is the best approximation in the following sense: For all functions $f(z)$ such that

$$(4.9) \qquad \int\limits_0^{2\pi} |f^{(m)}(z)|^2 d\theta \leqq H^2 \quad ,$$

where H is a fixed constant, we have the inequality

$$(4.10) \qquad \left| \int\limits_U f(z)dz - \lambda_* A \right| \leqq J_* \cdot H \quad , \qquad J_* = \frac{1}{m!}(2\pi(1-\lambda_*))^{\frac{1}{2}} \quad .$$

For any other inequality

$$\left| \int\limits_U f(z)dz - \sum_0^{k-1} C_j f(\omega^j) \right| \leqq J \cdot H$$

valid for the class (4.9), we must have $J > J_*$.

5. <u>The quadrature formula that is exact if</u> $f \in \mathbf{S}_{m,k}$. We wish to determine the quadrature formula

(5.1)
$$\int_U f(z)dz = \sum_0^{k-1} \widetilde{C}_j f(\omega^j) + \widetilde{R}f$$

such that

(5.2)
$$\widetilde{R}f = 0 \quad \text{if} \quad f \in \mathbf{S}_{m,k} \quad .$$

One way to construct such a formula is as follows. We assume that the class $\mathbf{S}_{m,k}$ has the interpolatory property at the knots ω^j . By [4, Theorem 1] we must

(5.3)   <u>exclude the case that</u> k <u>is even and</u> m <u>is odd.</u>

We are then assured [4] of the existence of a fundamental function $L_m(z) \in \mathbf{S}_{m,k}$ such that

(5.4)
$$f(z) = \sum_0^{k-1} f(\omega^j) L_m(z\omega^{-j}) \quad \text{if} \quad f \in \mathbf{S}_{m,k} \quad .$$

Integrating both sides of this identity along U we see that (5.1) has the property (5.2) if we set

(5.5)
$$\widetilde{C}_j = \int_U L_m(z\omega^{-j})dz \quad .$$

Again we assume (5.3). A second way of constructing this quadrature formula is as follows. Let $S_0(t)$ be the unique spline function such that the monospline

(5.6)
$$K_0(z) = \frac{1}{m!} z^m - S_0(z)$$

has the property

(5.7)
$$K_0(\omega^j) = 0 \quad (j = 0, \ldots, k-1) \quad .$$

The existence and unicity of $S_0(z)$ follow from [4, Theorem 1]. We shall now establish several properties of $K_0(z)$ which will show that this monospline will produce the quadrature formula we are looking for.

I. __The monospline (5.6) satisfies__

$$(5.8) \qquad\qquad K_0(z\omega) = \omega^m K_0(z) \quad .$$

__Proof:__ Clearly

$$\omega^{-m} K_0(z\omega) = \frac{1}{m!} z^m - \omega^{-m} S_0(z\omega)$$

is a monospline satisfying the relations (5.7). The unicity of $K_0(z)$ implies the relation (5.8).

II. __The quadrature formula__

$$(5.9) \qquad\qquad \int_U f(z)dz = \sum_0^{k-1} C_j^0 f(\omega^j) + R_0 f \quad ,$$

__where__ $- C_j^0 =$ __the jump of__ $K_0^{(m-1)}(z)$ __at__ $z = \omega^j$, __is exact if__ $f \in \mathbf{S}_{m,k}$.

__Proof:__ By Lemma 1 we may write

$$\pm R_0 f = \int_U K_0(z) f^{(m)}(z) dz = \int_U K_0(z) df^{(m-1)}(z) \quad \text{if} \quad f \in C^m(U) \quad .$$

However, the last form

$$(5.10) \qquad\qquad \pm R_0 f = \int_U K_0(z) df^{(m-1)}(z)$$

is also applicable (see proof of Lemma 1) if we only assume $f^{(m-2)}(z)$ to be absolutely continuous and $f^{(m-1)}(z)$ __of bounded variation on__ U . We may therefore assume in (5.9), (5.10) that $f(z) \in \mathbf{S}_{m,k}$ and, therefore, that $f^{(m-1)}(z)$ is a step function. If we denote by δ_j its jump at $z = \omega^j$, then (5.7) and (5.10) show that

$$\pm R_0 f = \sum K_0(\omega^j) \delta_j = 0$$

III.
$$C_j^0 = \widetilde{C}_j \qquad (j = 0, \ldots, k-1) \quad .$$

Proof: Letting $f(z) = L_m(z \omega^{-j})$ in (5.9), then (5.5) shows that

$$C_j^0 = \int_U L_m(z\omega^{-j})dz = \widetilde{C}_j \quad .$$

The explicit construction of the quadrature formula (5.9) now presents

no difficulties. We conclude from the relation (5.8) that $S_0(z)$ in an

m-flower and Lemma 3 implies that $K_0(z)$ may be written as

$$(5.11) \qquad K_0(z) = \frac{1}{m!} z^m - \lambda_0 \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2)\ldots(ks+m)} z^{ks+m}$$

for an appropriate value of the constant λ_0 . The defining properties (5.7),

in particular $K_0(1) = 0$ show that

$$(5.12) \qquad \lambda_0 = \left(\sum_{s=-\infty}^{\infty} \frac{m!}{(ks+1)(ks+2)\ldots(ks+m)} \right)^{-1} \quad .$$

This requires that the sum of the series in (5.12) should not vanish. By

[4, Lemma 4] this series can vanish only if $m = k-1$ and m is odd. But

this would imply that k is even and m is odd, a situation which can not occur

because of our assumption (5.3) .

We summarize our results in

Theorem 3. __If we exclude the case that__

$$(5.13) \qquad \qquad m = k-1 \quad \underline{and} \quad m \quad \underline{is \; odd} \; ,$$

then

$$(5.14) \qquad \int_U f(z)dz = \lambda_0 \, 2\pi i \, k^{-1} \sum_0^{k-1} \omega^j f(\omega^j) + (-1)^m \int_U K_0(z) f^{(m)}(z)dz$$

__is a quadrature formula that is exact if__ $f(z) \in \mathcal{S}_{m,k}$, __where__ $K_0(z)$ __and__ λ_0

__are described by__ (5.11) __and__ (5.12) . __If we replace__ (5.13) __by the more stringent__

__assumption__ (5.3) __then the quadrature formula__ (5.14) __may also be obtained by__

<u>integration from the spline interpolation formula</u> (5.4) .

The value $C_0 = 2\pi i \lambda_0 / k$ used in writing the formula (5.14) follow from (4.6) on replacing λ_* by λ_0 .

6. <u>The polynomial components of a certain spline function.</u> The spline function

$$(6.1) \qquad S_m(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2)\ldots(ks+m)} z^{ks+m}$$

appears in the representation (3.14), (3.15) of $K_*(z)$ as well as in (5.1!) which describes $K_0(z)$. By Lemma 3 it may also be characterized as an m-flower of degree m-1 . Here we propose to determine explicitly its individual polynomial components. Without loss of generality we may assume that $m = k-1$ and determine the components of

$$(6.2) \qquad S_{k-1}(z) = \sum_{-\infty}^{\infty} \frac{1}{(ks+1)\ldots(ks+k-1)} z^{ks+k-1}$$

it being clear by differentiations that

$$(6.3) \qquad S_m(z) = S_{k-1}^{(k-m-1)}(z) , \qquad (1 \leqq m \leqq k-1) \quad .$$

The functional equation

$$(6.4) \qquad S_{k-1}(z\omega) = \omega^{-1} S_{k-1}(z)$$

shows that it suffices to determine the polynomial component in the arc $(1, \omega)$. Setting

$$(6.5) \qquad \psi = \exp(\pi i / k)$$

we turn $S_{k-1}(z)$ by ψ^{-1} to obtain, since $\psi^k = -1$, the new spline function

$$(6.6) \qquad s(z) = \psi S(z\psi) = \sum_{-\infty}^{\infty} \frac{(-1)^{s+1}}{(ks+1)\ldots(ks+k-1)} z^{ks+k-1}$$

having its knots in the new locations $\psi \omega^j$. Let

$$(6.7) \qquad P_0(z) = \alpha_0 \frac{z^{k-2}}{(k-2)!} + \alpha_1 \frac{z^{k-3}}{(k-3)!} + \cdots + \alpha_{k-2}$$

be the component of $s(z)$ in the arc (ψ^{-1}, ψ) . The Laurent series (6.6) having real coefficients it is clear that $\overline{P(z)} = P(\bar{z})$ if z is on the arc (ψ^{-1}, ψ) and therefore also for all z . This shows that $P_0(z)$ is a <u>real</u> polynomial. Also $s(z)$ satisfies the relation $s(z\omega) = \omega^{-1} s(z)$ whence

$$(6.8) \qquad s(z) = \omega^{-1} s(z\omega^{-1}) \quad .$$

If z is in the arc $(\psi, \psi\omega)$, then (6.8) and (6.7) show that the component of $s(z)$ on this arc is

$$(6.9) \qquad P_1(z) = \alpha_0 \, \omega \, \frac{z^{k-2}}{(k-2)!} + \alpha_1 \omega^2 \, \frac{z^{k-3}}{(k-3)!} + \cdots + \alpha_{k-2} \omega^{k-1} \quad .$$

At this point we use that $s(z) \in C^{k-3}(U)$ and obtain at $z = \psi$ the $k-2$ equations

$$(6.10) \qquad P_0^{(\nu)}(\psi) = P_1^{(\nu)}(\psi) \qquad (\nu = 0, \ldots, k-3) \quad .$$

Using (6.7) and (6.9) we obtain the system

$$\alpha_0 (\omega-1) \frac{\psi^\nu}{\nu!} + \alpha_1 (\omega^2-1) \frac{\psi^{\nu-1}}{(\nu-1)!} + \cdots + \alpha_\nu (\omega^{\nu+1}-1) = 0 \qquad (\nu = 1, \ldots, k-2) \quad .$$

Dividing by $\omega^{\nu+1}-1$ and using the relations

$$\frac{\omega^\alpha-1}{\omega^\beta-1} \cdot \frac{\psi^{\beta-\alpha}}{(\beta-\alpha)!} = \frac{1}{(\beta-\alpha)!} \frac{\psi^\alpha-\psi^{-\alpha}}{\psi^\beta-\psi^{-\beta}} = \frac{1}{(\beta-\alpha)!} \frac{\sin(\pi\alpha/k)}{\sin(\pi\beta/k)}$$

we obtain the system

$$0 = \frac{1}{1!} \alpha_0 \sin(\pi/k) + \alpha_1 \sin(2\pi/k)$$

$$0 = \frac{1}{2!} \alpha_0 \sin(\pi/k) + \frac{1}{1!} \alpha_1 \sin(2\pi/k) + \alpha_2 \sin(3\pi/k)$$

$$\cdot$$
$$\cdot$$

$$0 = \frac{1}{(k-2)!} \alpha_0 \sin(\pi/k) + \frac{1}{(k-3)!} \alpha_1 \sin(2\pi/k) + \cdots + \alpha_{k-2} \sin((k-1)\pi/k)$$

which in terms of the new unknowns

(6.11) $$\beta_\nu = \alpha_\nu \sin \frac{(\nu+1)\pi}{k} \qquad (\nu = 0, \ldots, k-2)$$

becomes

$$0 = \frac{1}{1!}\,\beta_0 + \beta_1$$

$$0 = \frac{1}{2!}\,\beta_0 + \frac{1}{1!}\,\beta_1 + \beta_2$$

$$\vdots$$

$$0 = \frac{1}{(k-2)!}\,\beta_0 + \frac{1}{(k-3)!}\,\beta_1 + \cdots + \frac{1}{1!}\,\beta_{k-3} + \beta_{k-2} \quad .$$

This being the system giving the coefficients of the power series reciprocal

to $e^x = \sum x^\nu / \nu!$ we obtain a solution $\beta_\nu = (-1)^\nu / \nu!$ and by (6.11)

(6.12) $$\alpha_\nu = (-1)^\nu \Big/ \left(\nu! \sin \frac{(\nu+1)\pi}{k}\right), \qquad (\nu = 0, \ldots, k-2) \quad .$$

From (6.6) and (6.7) we therefore obtain the identity

(6.13) $$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\ldots(ks+k-1)}\, z^{ks+k-1} = c \cdot \sum_{\nu=0}^{k-2} (-1)^\nu \binom{k-2}{\nu} z^{k-\nu-2} \Big/ \sin \frac{(\nu+1)\pi}{k}$$

valid in the arc (ψ^{-1}, ψ) , while c is a constant yet to be determined.

To obtain c we set $z = 1$ to obtain

(6.14) $$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\ldots(ks+k-1)} = c \cdot \sum_{0}^{k-2} (-1)^\nu \binom{k-2}{\nu} \Big/ \sin \frac{(\nu+1)\pi}{k}$$

However, the left side is easily evaluated directly from the partial fraction

decomposition

(6.15) $$\frac{1}{(ks+1)\ldots(ks+k-1)} = \frac{1}{(k-2)!} \left\{ \frac{1}{(ks+1)} - \binom{k-2}{1}\frac{1}{ks+2} + \ldots + (-1)^{k-2}\frac{1}{ks+k-1} \right\}$$

From the classical expansion $\sum (-1)^s/(s+x) = \pi/\sin \pi x$ we obtain

$$\sum \frac{(-1)^s}{ks+\nu+1} = \frac{\pi}{k \sin \frac{(\nu+1)\pi}{k}}$$

If we multiply (6.15) by $(-1)^s$ and sum over all s we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\dots(ks+k-1)} = \frac{\pi}{(k-2)!k} \sum_{0}^{k-2} (-1)^\nu \binom{k-2}{\nu} \Big/ \sin \frac{(\nu+1)\pi}{k} \quad .$$

Comparing with (6.14) we find that

(6.16)
$$c = \frac{\pi}{(k-2)!k} \quad .$$

Using the relation (6.3) we can also determine the polynomial component of (6.1) __within the arc__ $(1, \omega)$. Indeed

$$\psi^{-m} S_m(z\psi) = \sum_s \frac{(-1)^s}{(ks+1)\dots(ks+m)} z^{ks+m}$$

and differentiating (6.13) $k-m-1$ times we obtain that

(6.17)
$$\psi^{-m} S_m(z\psi) = \frac{\pi}{k(m-1)!} \sum_{\nu=0}^{m-1} (-1)^\nu \binom{m-1}{\nu} z^{m-\nu-1} \Big/ \sin \frac{(\nu+1)\pi}{k}$$

if z is confined to the arc (ψ^{-1}, ψ) .

As an example let us determine $K_*(z)$ using the formulae (3.14), (3.15) for $k = 5$ and $m = 2$. The rapidly convergent series (3.13) shows that $\lambda_* = .969690$, and (6.17) furnishes the expression

$$\psi^{-2} K_*(z\psi) = \tfrac{1}{2} z^2 - \lambda_* \frac{\pi}{5 \sin(\pi/5)} \left(z - \frac{1}{2 \cos \frac{\pi}{5}} \right)$$

in the arc (ψ^{-1}, ψ) . Graphing the image of the arc (ψ^{-1}, ψ) by the quadratic polynomial on the right side and turning it by $\psi^2 = \omega$ (or $+72^\circ$) we obtain the image of the arc $(1, \omega)$ by $w = K_*(z)$. The functional equation

$$K_*(z\omega) = \omega^2 K_*(z)$$

allows to complete the image of the entire circle U which is shown in Fig. 1. The five corners of the curve are the images of the ω^j , Fig. 1 showing in parentheses the corresponding ω^j . Moreover

$$|K(\omega^j)| = .14063 , \qquad |K(\psi\omega^j)| = .10409 .$$

A general discussion and determination of the monosplines $K(z)$ of least L_∞ - norm would be of interest.
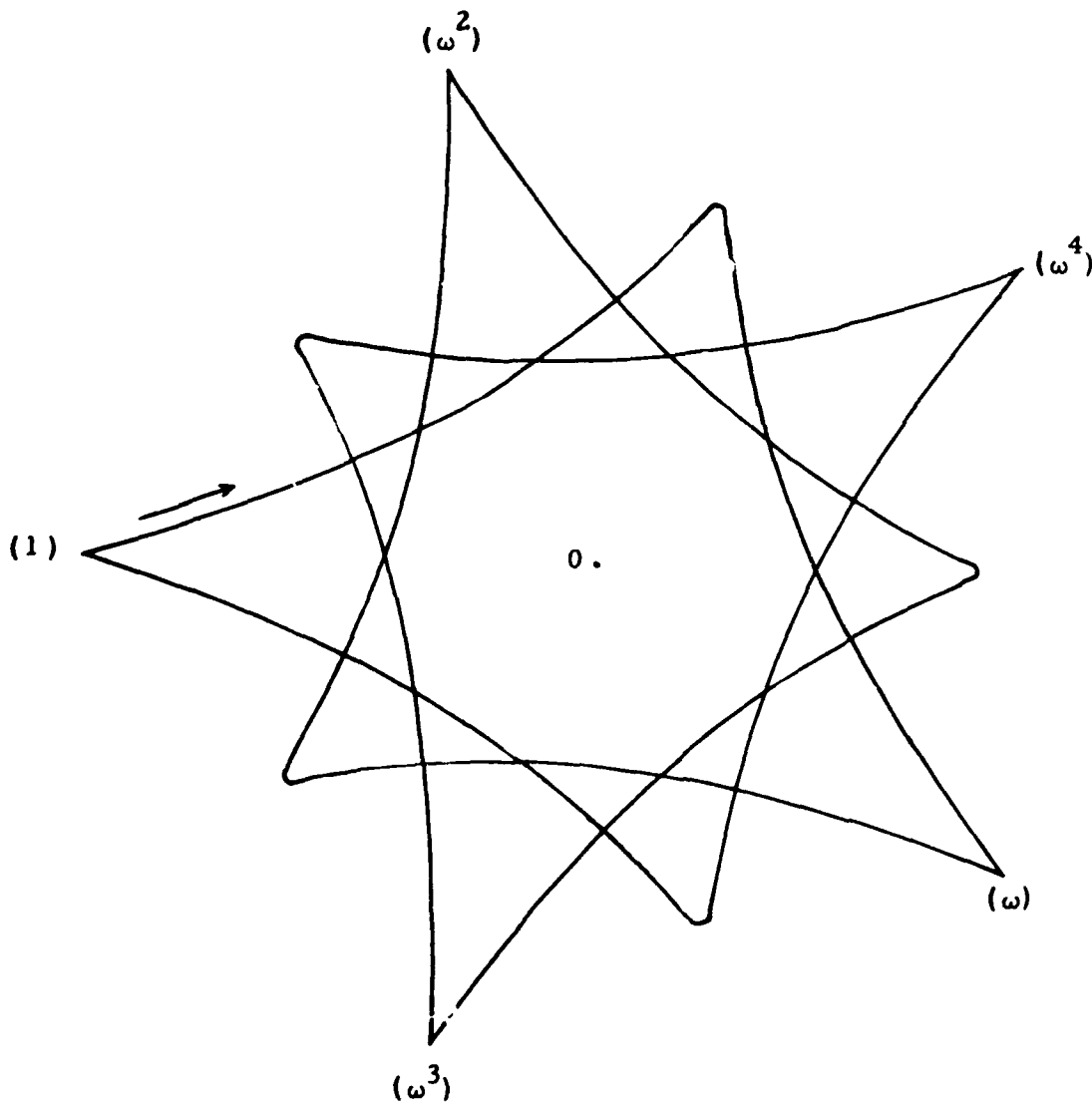

Fig. 1

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